Transformations and Invariants Connected with Linear Homogeneous Difference Equations and Other Functional Equations.[*]

By Cora B. Hennel.

Introduction.

The subject of Difference Equations has had an extended development during the last few years. Contributions[†] of importance have been made by Guichard, Nörlund, Galbrun, Carmichael, Birkhoff, Horn, Ford, Perron, Bôcher, and others.

The object of the present paper is to discuss for the difference equation primarily (Part I), and also for a general type of functional equation (Part II), the question of functions that remain invariant for a certain broad type of transformations. The results assume a very simple and elegant form.

In § 1, it is shown that the most general point transformation that changes every linear homogeneous difference equation into a difference equation that is linear, homogeneous, and of the same order as the given equation, is of the form

$$x = u(\xi), \quad y = \lambda(\xi)\, \eta_\xi, \tag{I}$$

where $u(\xi)$ satisfies one of the two relations

$$u(\xi + 1) - u(\xi) = \pm 1.$$

In § 2, the group character of the transformations (I) is verified and the existence of a certain subgroup is shown.

In § 3, a list of definitions of terms used in the remaining part of the paper is given.

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† See: Guichard, *Ann. de l'Éc. Norm.*, ser. (3), Vol. IV (1887), pp. 361–380; Nörlund, *Mém. Acad. Roy. Sc. et Let. de Danemark*, ser. (7), Vol. VI (1911), pp. 309–326; Galbrun, Thesis, Upsala, 1912, pp. 1–68; Carmichael, *Trans. Am. Math. Soc.*, Vol. XII (1911), pp. 99–134; Birkhoff, *Trans. Am. Math. Soc.*, Vol. XII (1911), pp. 243–284; Horn, *Crelle's Journal*, Vol. CXXXVIII (1910), pp. 159–191; Ford, *Trans. Am. Math. Soc.*, Vol. X (1909), pp. 319–336; Perron, *Acta Math.*, Vol. XXXIV (1911), pp. 109–137; Bôcher, *Ann. of Math.*, Vol. XVIII (1911), pp. 71–88. Further references will be found in the papers here referred to.

In §§ 4–7, certain fundamental sets of seminvariants, invariants, semi-covariants and covariants are determined.

In §§ 8–9, a certain general type of functional equation is dealt with, the procedure being along the same lines as that in the discussion of the difference equation in the preceding articles. In § 8, the most general point transformation that changes every linear homogeneous functional equation of the type under consideration into another of the same type and order is determined, the group character of these transformations is verified, and the existence of a certain subgroup is shown. In § 9, fundamental sets of seminvariants, invariants, semi-covariants and covariants are determined.

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PART I. TRANSFORMATIONS AND INVARIANTS CONNECTED WITH LINEAR HOMOGENEOUS DIFFERENCE EQUATIONS.

§ 1. *Determination of Point Transformations Leaving Certain Properties of Linear Homogeneous Difference Equations Invariant.*

We consider the totality of linear homogeneous difference equations. As a type of such equations, we use*

$$y_{x+n}+p_1(x)\,y_{x+n-1}+p_2(x)\,y_{x+n-2}+\ldots\ldots+p_{n-1}(x)\,y_{x+1}+p_n(x)\,y_x=0, \quad \text{(A)}$$

where $p_1(x), \ldots, p_n(x)$ are functions of x, arbitrary except that $p_n(x)$ is not identically zero, a restriction made necessary by the fact that, if $p_n(x)$ is not different from zero, the equation reduces at once to an equation of order less than n.

If x and y are regarded, for the moment, as coördinates of a point in two-dimensional space, then the most general so-called point transformation is of the form

$$x=\bar{u}(\xi, \eta_\xi), \quad y=\bar{v}(\xi, \eta_\xi).$$

We confine our attention to transformations of this type and equations of type (A), and determine first the most general transformation that transforms every equation (A) into another of the same type and order. That is, we determine the most general point transformation which leaves the following properties of equation (A) invariant:

* The two forms of functional notation are used to distinguish the dependent variable as a function of the independent variable from other functions of the independent variable that enter into the equation. This notation is used in general throughout the paper.

1) Its character of being a difference equation with argument difference unity;
2) Its degree;
3) Its linearity;
4) Its homogeneity.

For the sake of simplicity, let us consider first an equation of order one:

$$y_{x+1} + p_1(x)\, y_x = 0. \tag{1}$$

By the substitution

$$\left.\begin{aligned} x &= \bar{u}\,(\xi, \eta_\xi), \\ y_x &= \bar{v}\,(\xi, \eta_\xi), \end{aligned}\right\} \tag{2}$$

this equation becomes

$$\bar{v}\,[\phi(\xi), \eta_{\phi(\xi)}] + p_1\,[\bar{u}\,(\xi, \eta_\xi)]\,\bar{v}\,(\xi, \eta_\xi) = 0, \tag{3}$$

where $\phi(\xi)$ denotes what ξ becomes when x is replaced by $x+1$. We proceed to find what restrictions must be made on (2) in order that equation (3) may be a linear homogeneous difference equation of order one.

In the first place, equation (3) is to be linear in η_ξ. To meet this requirement, two conditions must be fulfilled. First, $\bar{u}(\xi, \eta_\xi)$ must be a function of ξ alone; for otherwise, since $p_1(x)$ is arbitrary, $p_1(x)$ can be chosen so that $p_1[\bar{u}(\xi, \eta_\xi)]$ shall be non-linear in η_ξ. Secondly, $\bar{v}(\xi, \eta_\xi)$ must be linear in η_ξ. The transformation, then, must be of the form

$$\begin{aligned} x &= u\,(\xi), \\ y_x &= \lambda\,(\xi)\,\eta_\xi + \theta\,(\xi), \end{aligned}$$

and equation (3) takes the form

$$\lambda[\phi(\xi)]\,\eta_{\phi(\xi)} + \theta\,[\phi(\xi)] + p_1[u(\xi)]\,\lambda(\xi)\,\eta_\xi + p_1[u(\xi)]\,\theta(\xi) = 0.$$

Now this equation is to be homogeneous. Obviously, a necessary and sufficient condition for this is that $\theta(\xi)$ shall satisfy the equation

$$\theta[\phi(\xi)] + p_1[u(\xi)]\,\theta(\xi) = 0;$$

since p_1 is arbitrary, it follows that $\theta(\xi)$ must be zero.

Finally, the transformed equation, which now has the form

$$\lambda[\phi(\xi)]\,\eta_{\phi(\xi)} + p_1[u(\xi)]\,\lambda(\xi)\,\eta_\xi = 0,$$

is to be a difference equation of order one. That this condition may be fulfilled, one of the two equations

$$\phi\,(\xi) = \xi \pm 1 \tag{4}$$

must hold.

This last restriction says that as x is changed into $x+1$, ξ becomes either $\xi+1$ or $\xi-1$; it also imposes a restriction on $u(\xi)$, which we shall now determine.

From the relation $x=u(\xi)$, existing between x and ξ, we obtain, at once,

$$\xi = u^{-1}(x).$$

Let us now replace x by $x+1$, and then substitute for x its value $u(\xi)$; thus:

$$\phi(\xi) = u^{-1}(x+1) = u^{-1}[u(\xi)+1].$$

Using the latter value for $\phi(\xi)$ in equations (4), we obtain

$$u^{-1}[u(\xi)+1] = \xi \pm 1,$$

which is equivalent to the two equations

$$u(\xi)+1 = u(\xi+1),$$
$$u(\xi)+1 = u(\xi-1).$$

From the first of these two, we obtain, by transposing,

$$u(\xi+1) - u(\xi) = 1;$$

from the second, we obtain, by replacing ξ by $\xi+1$, and transposing,

$$u(\xi+1) - u(\xi) = -1.$$

That is, $u(\xi)$ must satisfy one of the two equations

$$u(\xi+1) - u(\xi) = \pm 1.$$

We have shown, then, that the most general point transformation that changes every linear homogeneous difference equation of the first order into another of the same type and order, is of the form

$$x = u(\xi),$$
$$y_x = \lambda(\xi)\eta_\xi,$$

where $\lambda(\xi)$ is arbitrary, and $u(\xi)$ satisfies one of the relations

$$u(\xi+1) - u(\xi) = \pm 1,$$

but is otherwise arbitrary.

Consider now the n-th order equation (A). Since the transformation sought must leave invariant properties 1) to 4) of equations of form (A) of any degree, the restrictions on the transformation found necessary for an equation of degree one are necessary also in the case of equations of degree n. That these restrictions are sufficient, as well, is readily seen.

By a transformation of the form

$$x = u(\xi), \quad y_x = \lambda(\xi)\eta_\xi,$$

equation (A) becomes

$$\eta_{\xi+n}+p_1[u(\xi)]\,\frac{\lambda(\xi+n-1)}{\lambda(\xi+n)}\,\eta_{\xi+n-1}+\ldots+p_n[u(\xi)]\,\frac{\lambda(\xi)}{\lambda(\xi+n)}\,\eta_\xi=0$$

or

$$\eta_{\xi-n}+p_1[u(\xi)]\,\frac{\lambda(\xi-\overline{n-1})}{\lambda(\xi-n)}\,\eta_{\xi-\overline{n-1}}+\ldots+p_n[u(\xi)]\,\frac{\lambda(\xi)}{\lambda(\xi-n)}\,\eta_\xi=0,$$

according as $u(\xi)$ satisfies the relation

$$u(\xi+1)-u(\xi)=1$$

or

$$u(\xi+1)-u(\xi)=-1.$$

Obviously, in either case the transformed equation is a linear homogeneous equation of order n.

This completes the proof of the following theorem:

THEOREM I. *The most general transformation of the form*

$$x=\bar{u}(\xi,\eta_\xi),$$
$$y_x=\bar{v}(\xi,\eta_\xi)$$

that transforms every linear homogeneous difference equation of the form

$$y_{x+n}+p_1(x)\,y_{x+n-1}+\ldots+p_n(x)\,y_x=0 \tag{A}$$

into a difference equation that is linear, homogeneous, and of the same order as the given equation, is

$$\left.\begin{aligned} x&=u(\xi),\\ y_x&=\lambda(\xi)\,\eta_\xi, \end{aligned}\right\}\quad \text{(I)*}$$

where $u(\xi)$ satisfies one of the two relations

$$u(\xi+1)-u(\xi)=\pm 1,$$

and $\lambda(\xi)$ is arbitrary.

§ 2. *On a Subgroup of the Group of Transformations.*

We have proved that any transformation of type (I) transforms every equation (A) into a difference equation that is linear, homogeneous and of order n, and that any point transformation that does this is of type (I). Obviously, then, the transformations (I) form a group. For we see at once—or can readily show—that any two transformations of type (I) can be replaced by a single transformation that gives the same resulting equation; this latter

* Throughout the discussion we refer to the transformation

$$x=u(\xi),\quad y_x=\lambda(\xi)\,\eta_\xi$$

as (I′) or (I″), according as $u(\xi)$ satisfies the relation $u(\xi+1)-u(\xi)=1$ or $u(\xi+1)-u(\xi)=-1$. If $u(\xi)$ satisfies merely one or the other, indiscriminately, we refer to the transformation as (I).

transformation, since it changes (A) into a difference equation that is linear, homogeneous and of order n, must be of type (I).

The transformations (I′) form a subgroup of the total group (I), while the transformations (I″) do not, as we shall now show.

Consider two transformations of type (I′):

$$x = u'(\xi'), \qquad u'(\xi'+1) - u'(\xi') = 1,$$
$$y_x = \lambda'(\xi')\,\eta'_{\xi'},$$

and

$$\xi' = u''(\xi''), \qquad u''(\xi''+1) - u''(\xi'') = 1,$$
$$\eta'_{\xi'} = \lambda''(\xi'')\,\eta''_{\xi''}.$$

Combining these transformations, we have

$$\left.\begin{aligned} x &= u'[u''(\xi'')], \\ y_x &= \lambda'[u''(\xi'')]\,\lambda''(\xi'')\,\eta''_{\xi''}, \end{aligned}\right\} \tag{5a}$$

with the restrictions

$$u'[u''(\xi'')+1] - u'[u''(\xi'')] = 1,$$
$$u''(\xi''+1) - u''(\xi'') = 1.$$

The first of these restrictions comes from that on $u'(\xi')$ in the first transformation, taken with the relation between ξ and ξ''; the second is obviously that on $u''(\xi'')$ in the second transformation.

Replacing $u''(\xi'')+1$ in the first of these equations by its equivalent $u''(\xi''+1)$ obtained from the second, we have

$$u'[u''(\xi''+1)] - u'[u''(\xi'')] = 1.$$

Now this is precisely the restriction that $u'[u''(\xi'')]$ must satisfy in order that transformation (5a) may be of type (I′). Hence the transformations (I′) form a subgroup of (I).

To show that the transformations (I″) do not form a subgroup, let us use the same two transformations, except that we shall now take as the restrictions on $u'(\xi')$ and $u''(\xi'')$ the following:

$$u'(\xi'+1) - u'(\xi') = -1,$$
$$u''(\xi''+1) - u''(\xi'') = -1.$$

Combining the transformations, we get

$$\left.\begin{aligned} x &= u'[u''(\xi'')], \\ y_x &= \lambda'[u''(\xi'')]\,\lambda''(\xi'')\,\eta''_{\xi''}, \end{aligned}\right\} \tag{5b}$$

with the restrictions

$$u'[u''(\xi'')+1] - u'[u''(\xi'')] = -1,$$
$$u''(\xi''+1) - u''(\xi'') = -1.$$

If in the first of these restrictions we substitute for $u''(\xi'')+1$ its equivalent obtained from the second by replacing ξ by $\xi-1$ and transposing, we obtain:

$$u'[u''(\xi''-1)]-u'[u''(\xi'')]=-1;$$

whence, putting $\xi''+1$ for ξ'', and transposing:

$$u'[u''(\xi''+1)]-u'[u''(\xi'')]=+1.$$

This result shows that while transformation (5b) is indeed of type (I), it is of type (I′) and not (I″). Hence the transformations (I″) do not form a subgroup.

We have shown, incidentally, that the successive operation of two transformations of form (I′) or of form (I″) is equivalent to a single transformation of type (I). Similarly, we might show that the combination of any transformation of form (I′) with one of (I″) is also equivalent to a single transformation of form (I), thus completely verifying the fact that the transformations (I) do indeed form a group.

We sum up these results in the form of a theorem, as follows:

THEOREM II. *The transformations* (I) *form a group, of which the transformations* (I′) *form a subgroup. The transformations* (I″) *do not form a subgroup.*

§ 3. *Definitions.*

We state now definitions of certain terms used in the sections that follow. Terms used in the field of differential equations are given analogous meanings here.

Any equation which can be obtained from equation (A) by a transformation of form (I) is said to be *equivalent* to (A).

An *absolute invariant*, or more briefly, an *invariant*, is any function of the p's and their differences,

$$F(p_1, p_2, \ldots, p_n;\ \Delta p_1, \ldots, \Delta p_n;\ \Delta^2 p_1, \ldots;\ \ldots),$$

which is equal to the same function formed for an equivalent equation.

A *seminvariant* is any function of the p's and their differences which is equal to the same function formed for an equation derived from (A) by a transformation of the form

$$x=x, \quad y_x=\lambda(x)\,\eta_x.$$

A number of invariants [seminvariants], $\alpha_1, \alpha_2, \ldots$, is said to be a *fundamental set* of invariants [seminvariants] if every invariant [seminvariant]

can be expressed as a function of the α's and their differences, and no α can be expressed as a function of the remaining α's and their differences.

A *covariant* is any function of the p's and their differences, and y and its differences, which is equal to the same function formed for an equivalent equation.

A *semi-covariant* is any function of the p's and their differences, and y and its differences, which is equal to the same function formed for an equation derived from (A) by a transformation of the form

$$x = x, \quad y_x = \lambda(x)\,\eta_x.$$

A number of covariants [semi-covariants], $\beta_1, \beta_2, \ldots.,$ is said to be a *fundamental set* of covariants [semi-covariants] if every covariant [semi-covariant] can be expressed as a function of the β's and their differences, and invariants [seminvariants] and their differences, and no β can be expressed in terms of the remaining β's and their differences and invariants [seminvariants] and their differences.

Let

$$\bar{p}_0(\xi)\eta_{\xi\pm n} + \bar{p}_1(\xi)\eta_{\xi\pm\overline{n-1}} + \ldots. + \bar{p}_{n-1}(\xi)\eta_{\xi\pm 1} + \bar{p}_n(\xi)\eta_\xi = 0$$

be any equation equivalent to (A). This equation is said to be a normal form of (A) provided either of the following conditions holds:

$$\begin{aligned} &1) \quad \bar{p}_0(\xi) \equiv \bar{p}_1(\xi) \equiv 1, \\ &2) \quad \bar{p}_{n-1}(\xi) \equiv \bar{p}_n(\xi) \equiv 1. \end{aligned}$$

§ 4. *Fundamental Sets of Seminvariants.*

We determine now certain fundamental sets of seminvariants.

If we make the substitution

$$x = x, \quad y_x = \lambda(x)\,\eta_x \tag{6}$$

in equation (A),

$$y_{x+n} + p_1(x)\,y_{x+n-1} + \ldots. + p_n(x)\,y_x = 0, \tag{A}$$

we obtain, after dividing through by $\lambda(x+n)$,

$$\eta_{x+n} + p_1(x)\frac{\lambda(x+n-1)}{\lambda(x+n)}\eta_{x+n-1} + p_2(x)\frac{\lambda(x+n-2)}{\lambda(x+n)}\eta_{x+n-2} + \ldots. + p_n(x)\frac{\lambda(x)}{\lambda(x+n)}\eta_x = 0.$$

Let us reduce this to a normal form, by choosing λ so that the coefficient of η_{x+n-1} shall be unity:

$$\frac{\lambda(x+n-1)}{\lambda(x+n)}\,p_1(x) = 1.$$

Obviously it is possible to choose λ in this way, for the equation determining λ is a first-order difference equation which has a solution, provided $p_1(x)$ is not identically zero. We assume, in the determination of a first fundamental set of seminvariants, that $p_1(x)$ is not identically zero.

Dividing both sides of the last equation by $p_1(x)$, we have

$$\frac{\lambda(x+n-1)}{\lambda(x+n)}=\frac{1}{p_1(x)}. \tag{7}$$

Without actually solving this equation, we find what the coefficients

$$p_2(x)\frac{\lambda(x+n-2)}{\lambda(x+n)}, \;\ldots\ldots,\; p_n(x)\frac{\lambda(x)}{\lambda(x+n)}$$

become when λ satisfies the equation. Replacing x by $x-1$, in (7), and multiplying the resulting equation by (7), member by member, we obtain

$$\frac{\lambda(x+n-2)}{\lambda(x+n)}=\frac{1}{p_1(x)\,p_1(x-1)}.$$

Replacing x by $x-1$ in this equation, and multiplying the resulting equation by (7), we obtain

$$\frac{\lambda(x+n-3)}{\lambda(x+n)}=\frac{1}{p_1(x)\,p_1(x-1)\,p_1(x-2)}.$$

Obviously, a continuation of this process gives the result

$$\frac{\lambda(x+n-r)}{\lambda(x+n)}=\frac{1}{p_1(x)\,p_1(x-1)\ldots\ldots p_1(x-\overline{r-1})}, \quad (r=2,\ldots\ldots,n).$$

Hence a normal form of (A) is

$$\eta_{x+n}+\eta_{x+n-1}+P_2(x)\,\eta_{x+n-2}+\ldots\ldots+P_n(x)\,\eta_x=0, \tag{8}$$

where

$$P_r(x)=\frac{p_r(x)}{p_1(x)\,p_1(x-1)\ldots\ldots p_1(x-\overline{r-1})}.$$

The functions $P_r(x)$, $r=2,\ldots\ldots,n$, are themselves seminvariants, as we shall now show.

If we apply the transformation

$$x=x, \quad y_x=\lambda'(x)\,\eta'_x \tag{9}$$

to equation (A), we obtain

$$\eta'_{x+n}+p'_1(x)\,\eta'_{x+n-1}+\ldots\ldots+p'_n(x)\,\eta'_x=0, \tag{10}$$

where

$$p'_r(x)=p_r(x)\frac{\lambda'(x+n-r)}{\lambda'(x+n)}, \quad (r=1,\ldots\ldots,n).$$

We can reduce this to a normal form by making the further transformation

$$x = x, \quad \eta'_x = \lambda''(x)\,\eta_x, \tag{11}$$

where $\lambda''(x)$ satisfies the relation

$$\frac{\lambda''(x+n-1)}{\lambda''(x+n)} = \frac{1}{p'_1(x)}. \tag{12}$$

Let us take as the value of $\lambda''(x)$, $\dfrac{\lambda(x)}{\lambda'(x)}$. That this value of λ'' satisfies relation (12) is readily seen, by substituting it for λ'' in (12), making use of the value of $p'_r(x)$ given in (10):

$$\frac{\lambda(x+n-1)}{\lambda(x+n)} \cdot \frac{\lambda'(x+n)}{\lambda'(x+n-1)} = \frac{1}{p_1(x)} \cdot \frac{\lambda'(x+n)}{\lambda'(x+n-1)},$$

whence

$$\frac{\lambda(x+n-1)}{\lambda(x+n)} = \frac{1}{p_1(x)},$$

which is an identity. By means of transformation (11), equation (10) becomes

$$\eta_{x+n} + \eta_{x+n-1} + P'_2(x)\,\eta_{x+n-2} + \ldots\ldots + P'_n(x)\,\eta_x = 0, \tag{13}$$

where

$$P'_r(x) = \frac{p'_r(x)}{p'_1(x)\,p'_1(x-1)\ldots\ldots p'_1(x-r+1)}, \quad (r = 2, \ldots\ldots, n).$$

Now the transformation obtained by combining (9) and (11) is exactly the same as (6); hence equations (13) and (8) are identical. Consequently for any particular r, $P'_r(x)$ and $P_r(x)$ are the same. But $P'_r(x)$ is a function of the coefficients $p'_1, \ldots\ldots, p'_n$ of the equation derived from (A) by a transformation of form (6); since it is the same function of the coefficients p' that $P_r(x)$ is of the p's, $P_r(x)$ must be a seminvariant.

We have as seminvariants, then, the functions $P_2(x), \ldots\ldots, P_n(x)$. Obviously no P can be expressed in terms of the remaining P's and their differences, and any function of these seminvariants and their differences is a seminvariant. We show, finally, that every seminvariant can be expressed in terms of the P's and their differences.

Let

$$F_1(p_1, p_2, \ldots\ldots, p_n;\ \Delta p_1, \ldots\ldots;\ \ldots\ldots)$$

be a seminvariant. This function must by definition be equal to the same function formed for any equation derived from (A) by a transformation of type (6); it must in particular, therefore, be equal to that formed for the normal form (8):

$$F_1(1, P_2, \ldots\ldots, P_n;\ \Delta P_2, \ldots\ldots;\ \ldots\ldots);$$

that is, it must be a function of the seminvariants $P_2, \ldots, P_n$ and their differences. Hence the P's form a fundamental set of seminvariants.

We have now determined one fundamental set of seminvariants, on the assumption that $p_1(x)$ is not identically zero. On the assumption that $p_{n-1}(x)$ is not identically zero, we can readily determine another, by a method differing from that just employed only in that the second of the two normal forms of (A), instead of the first, is used.

To obtain the second normal form of (A), we first divide the equation through by $p_n(x)$ in order that the coefficient of y_x may be unity, denoting the resulting coefficients by $\pi_n(x), \ldots, \pi_1(x)$ respectively. We then choose λ so that the coefficient of η_{x+1} in the transformed equation (after the equation has been divided through by the coefficient of η_x) shall be unity. Using then, as we have already indicated, a method analogous to that employed in determining the first fundamental set of seminvariants, we find, as a second set, the functions $\Pi_r(x)$, $r = n, \ldots, 2$, where

$$\Pi_r(x) = \frac{\pi_r(x)}{\pi_1(x)\,\pi_1(x+1)\ldots\pi_1(x+\overline{r-1})}.$$

We have yet to consider the exceptional case in which both $p_1(x)$ and $p_{n-1}(x)$ are identically zero. We include this case in the following discussion, which covers every other case as well. Inasmuch as the two cases already discussed are but particular cases of the two general cases which we now consider, we might have omitted the separate discussion of them, taking up at once the general problem. On account of the importance of the two particular cases, however, and because the treatment of them first simplifies the proof for the general cases, it has seemed advisable to adopt the order here followed.

Let us suppose that $p_s(x)$ is the first of the coefficients in (A) (after the coefficient of η_{x+n}, which is unity) that is not identically zero. The equation then has the form

$$y_{x+n} + p_s(x)\,y_{x+n-s} + p_{s+1}(x)\,y_{x+n-\overline{s+1}} + \ldots \\ + p_{n-1}(x)\,y_{x+1} + p_n(x)\,y_x = 0. \qquad (A_1)$$

By means of the transformation

$$x = x, \quad y_x = \lambda(x)\,\eta_x,$$

where $\lambda(x)$ satisfies the relation

$$\frac{\lambda(x+n-s)}{\lambda(x+n)} = \frac{1}{p_s(x)},$$

we change (A_1) into a form in which the coefficient η_{x+n-s} is unity:

$$\eta_{x+n}+\eta_{x+n-s}+Q_{s+1}(x)\,\eta_{x+n-\overline{s+1}}+\ldots\ldots+Q_n(x)\,\eta_x=0, \qquad (14)$$

where

$$Q_r(x)=p_r(x)\,\frac{\lambda(x+n-r)}{\lambda(x+n)}, \quad (r=s+1,\ldots\ldots,n).$$

In the particular case in which s is unity (which we have already considered), the normal form (14) is unique and the coefficients $Q_r(x)$ are readily expressible in terms of the p's; for any value of s different from unity, the normal form (14) is not unique and the Q's can not be expressed as simple functions of the p's; hence a modification of the method used in the case $s=1$ is necessary.

Although the Q's themselves are not readily expressible in terms of the p's, there are simple combinations of them that are, and we now form a set of such combinations constituting a fundamental set of seminvariants. These functions, which we denote by $S_r(x)$, $r=s+1,\ldots\ldots,n$, are:*

$$S_r(x)=Q_r(x)\,Q_r(x-r)\ldots\ldots Q_r(x-\overline{s-1}\,r).$$

Let us express these in terms of the p's by means of relation (12):

$$S_r(x)=p_r(x)\,p_r(x-r)\ldots p_r(x-\overline{s-1}\,r)\cdot\frac{\lambda(x+n-r)}{\lambda(x+n)}$$
$$\cdot\frac{\lambda(x+n-2r)}{\lambda(x+n-r)}\cdots\frac{\lambda(x+n-sr)}{\lambda(x+n-\overline{r-1}s)}$$
$$=p_r(x)\ldots p_r(x-\overline{s-1}r)\cdot\frac{\lambda(x+n-rs)}{\lambda(x+n)}$$
$$=p_r(x)\ldots p_r(x-\overline{s-1}r)\cdot\frac{\lambda(x+n-s)}{\lambda(x+n)}\cdot\frac{\lambda(x+n-2s)}{\lambda(x+n-s)}\cdots\frac{\lambda(x+n-rs)}{\lambda(x+n-\overline{r-1}s)}$$
$$=\frac{p_r(x)\,p_r(x-r)\ldots p_r(x-\overline{s-1}\,r)}{p_s(x)\,p_s(x-s)\ldots p_s(x-\overline{r-1}s)}.$$

That the S's are seminvariants can be readily verified; that no S can be expressed in terms of the remaining S's and their differences is easily seen. To prove, then, that they form a fundamental set of seminvariants, we need merely show that any seminvariant can be expressed in terms of the S's and their differences. For this purpose let us consider the function

$$F[p_s(x),\ p_{s+1}(x),\ \ldots\ldots;\ \Delta p_s(x),\ \ldots\ldots;\ \ldots\ldots]$$

and show that a function G can be found satisfying the following relation:

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* The form of these functions was suggested by the formal process of obtaining from the seminvariants $P_2,\ldots\ldots,P_n$ functions independent of $p_1(x)$.

$$F[p_s(x), p_{s+1}(x), \ldots; \ldots] = G[p_s(x)\, p_s(x-s) \ldots p_s(x-\overline{s-1}\,s),$$
$$p_{s+1}(x) \ldots p_{s+1}(x-\overline{s-1}\,\overline{s+1}), \ldots; \ldots]. \quad (15)$$

A sufficient condition that a G exist, satisfying this equation, is that there exist functions H_r, $r = s, \ldots, n$, satisfying the relations

$$H_r[p_r(x)\, p_r(x-r) \ldots p_r(x-\overline{s-1}\,r)] = p_r(x). \quad (16)$$

If we write

$$p_r(x)\, p_r(x-r) \ldots p_r(x-\overline{s-1}\,r) = t,$$

whence we have

$$x = f(t),$$

the existence of such functions is obvious:

$$p_r(x) = p_r[f(t)] = H_r(t).$$

Now let F_1 be any invariant. Since F_1 is an invariant, it is equal to the same function formed for the normal form (14); that is, we have

$$F_1[p_s(x), p_{s+1}(x), \ldots; \ldots] = F_1[1, Q_{s+1}(x), \ldots].$$

But by (15), we have

$$F_1[1, Q_{s+1}(x), \ldots] = G_1[1, Q_{s+1}(x) \ldots Q_{s+1}(x-sr), \ldots]$$
$$= G_1[1, S_{s+1}(x), \ldots].$$

Hence any seminvariant F_1 can be expressed in terms of the functions $S_r(x)$ and their differences, and therefore the functions $S_r(x)$ form a fundamental set of seminvariants.

Similarly, on the supposition that all coefficients in (A) following $p_{n-m}(x)$ (except, of course, that of y_x, which can not be zero) are identically zero, we obtain as another fundamental set of seminvariants the following:

$$\Sigma_r(x) = \frac{\pi_r(x)\, \pi_r(x+r) \ldots \pi_r(x+\overline{m-1}\,r)}{\pi_m(x)\, \pi_m(x+m) \ldots \pi_m(x+\overline{r-1}\,m)}, \quad (r = n, \ldots, m+1).$$

We state these results in the form of a theorem as follows:

Theorem III. *If all the coefficients in* (A) *preceding* $p_s(x)$ (*except, of course, that of* y_{x+n}, *which is unity*) *are identically zero, the functions* $S_r(x)$, $r = s+1, \ldots, n$, *where*

$$S_r(x) = \frac{p_r(x)\, p_r(x-r) \ldots p_r(x-\overline{s-1}\,r)}{p_s(x)\, p_s(x-s) \ldots p_s(x-\overline{r-1}\,s)},$$

form a fundamental set of seminvariants.

If all the coefficients in (A) *following* $p_{n-m}(x)$ (*except, of course, that of* y_x, *which is different from zero*) *are identically zero, the functions* $\Sigma_r(x)$, $r=n, \ldots, m+1$, *where*

$$\Sigma_r(x)=\frac{\pi_r(x)\,\pi_r(x+r)\ldots\pi_r(x+\overline{m-1}\,r)}{\pi_m(x)\,\pi_m(x+m)\ldots\pi_m(x+\overline{r-1}\,m)}$$

and

$$\pi_i(x)=\frac{p_{n-i}(x)}{p_n(x)},\quad \pi_0(x)=\frac{1}{p_n(x)},$$

form a fundamental set of seminvariants.

In the important cases in which $s=1$ *and* $m=1$, *the functions* $S_r(x)$ *and* $\Sigma_r(x)$ *reduce to* $P_r(x)$ *and* $\Pi_r(x)$ *respectively, where*

$$P_r(x)=\frac{p_r(x)}{p_1(x)\,p_1(x-1)\ldots p_1(x-\overline{r-1})}$$

and

$$\Pi_r(x)=\frac{\pi_r(x)}{\pi_1(x)\,\pi_1(x+1)\ldots\pi_1(x+\overline{r-1})}.$$

§ 5. *Fundamental Sets of Invariants.*

By a method analogous to that used in the discussion of seminvariants, we now determine certain fundamental sets of invariants. In dealing with transformations of type (I), we must remember that transformations (I′) and (I″) give the transformed equation in different forms. (See page 435.) However, the methods for determining the invariants for the two transformations differ very little, and the results turn out to be the same. Accordingly we shall discuss only the transformation (I′) and assume the result for (I″).

Let us consider first the case in which $p_1(x)$ is not identically zero. We can reduce (A) to a normal form by means of the transformation

$$\left.\begin{aligned} &x=u(\xi),\qquad u(\xi+1)-u(\xi)=1,\\ &y_x=\lambda(\xi)\eta_\xi,\end{aligned}\right\}\quad(17)$$

where

$$\frac{\lambda(\xi+n-1)}{\lambda(\xi+n)}=\frac{1}{p_1[u(\xi)]}.$$

We obtain thus

$$\eta_{\xi+n}+\eta_{\xi+n-1}+P_2[u(\xi)]\,\eta_{\xi+n-2}+\ldots+P_n[u(\xi)]\,\eta_\xi=0,\qquad(18)$$

where

$$P_r[u(\xi)]=\frac{p_r[u(\xi)]}{p_1[u(\xi)]\,p_1[u(\xi-1)]\ldots p_1[u(\xi-r+1)]}.$$

The functions P_r form a fundamental set of invariants, as we shall now show.

Going back to equation (A) and transforming it by means of the transformation

$$\left.\begin{array}{ll} x = u'(\xi'), & u'(\xi'+1) - u'(\xi') = 1, \\ y_x = \lambda'(\xi')\eta'_{\xi'}, & \end{array}\right\} \quad (19)$$

we obtain

$$\eta'_{\xi'+n} + p'_1[u'(\xi')]\,\eta'_{\xi'+n-1} + \dots + p'_n[u'(\xi')]\,\eta'_{\xi'} = 0, \quad (20)$$

where

$$p'_r[u'(\xi')] = p_r[u'(\xi')]\,\frac{\lambda'(\xi'+n-r)}{\lambda'(\xi'+n)}, \quad (r = 1, \dots, n).$$

We change this equation to a form in which the coefficient of $\eta_{\xi''+n-1}$ is unity by means of the further transformation

$$\left.\begin{array}{ll} \xi' = u''(\xi''), & u''(\xi''+1) - u''(\xi'') = 1, \\ \eta'_{\xi'} = \lambda''(\xi'')\,\eta_{\xi''}, & \end{array}\right\} \quad (21)$$

where

$$\frac{\lambda''(\xi''+n-1)}{\lambda''(\xi''+n)} = \frac{1}{p'_1[u''(\xi'')]}$$

and

$$u'[u''(\xi'')] = u(\xi).$$

This restriction is placed on $u''(\xi)$ in order that the combination of transformations (19) and (21) may change x exactly as the single transformation (I′) does.

By means of (21) we get

$$\eta_{\xi''+n} + \eta_{\xi''+n-1} + P'_2\{u'[u''(\xi'')]\}\,\eta_{\xi''+n-2} + \dots + P'_n\{u'[u''(\xi'')]\}\,\eta_{\xi''} = 0, \quad (22)$$

where

$$P'_r\{u'[u''(\xi'')]\} = P'_r[u(\xi)] = \frac{p'_r\{u'[u''(\xi'')]\}}{p'_1\{u'[u''(\xi'')]\}\dots p'_1\{u'[u''(\xi''-r+1)]\}}.$$

Obviously, the functions P'_r are precisely the same functions of $p'_1, \dots, p'_n$ as the functions P_r are of $p_1, \dots, p_n$. But equations (18) and (22) are identical, as we shall now show, and hence

$$P'_r[u(\xi)] = P_r[u(\xi)];$$

that is, the functions $P_r[u(\xi)]$ are invariants.

To show that equations (18) and (22) are identical, we need merely consider the combination of transformations used to get (22):

$$\left.\begin{array}{ll} x = u'[u''(\xi'')], & u'[u''(\xi''+1)] - u'[u''(\xi'')] = 1, \\ y_x = \lambda'[u''(\xi'')]\,\lambda''(\xi'')\,\eta_{\xi''}, & \end{array}\right\} \quad (23)$$

and notice the following relations:

$$\frac{\lambda''(\xi''+n-1)}{\lambda''(\xi''+n)}=\frac{1}{p_1'[u'(\xi')]}=\frac{\lambda'(\xi''+n)}{\lambda'(\xi''+n-1)}\cdot\frac{1}{p_1'\{u'[u''(\xi'')]\}},$$

whence

$$\frac{\lambda''(\xi''+n-1)}{\lambda''(\xi''+n)}\cdot\frac{\lambda'(\xi''+n-1)}{\lambda'(\xi''+n)}=\frac{1}{p_1[u(\xi)]}=\frac{\lambda(\xi+n-1)}{\lambda(\xi+n)}.$$

This last relation shows that $\lambda'[u''(\xi'')]\,\lambda''(\xi'')$ can differ from $\lambda(\xi)$ only by a periodic multiplier, and hence transformations (17) and (23) give the same resulting equation. Hence P_r' and P_r are identical, for any particular r. But P_r' is a function of the coefficients $p_1', \ldots, p_n'$ of any equation derived from (A) by a transformation of type (I'); since it is the same function of the coefficients p' that P_r is of the p's, P_r must be an invariant.

We have as invariants, then, the functions $P_2[u(\xi)]$, $P_3[u(\xi)]$, $\ldots$, $P_n[u(\xi)]$. Obviously, no P can be expressed in terms of the remaining P's and their differences, and any function of these invariants and their differences is an invariant. We show, finally, that every invariant can be expressed in terms of the P's and their differences.

Let

$$F_1(p_1, p_2, \ldots, p_n;\ \Delta p_1, \ldots;\ \ldots)$$

be an invariant. This function must by definition be equal to the same function formed for any equation derived from (A) by a transformation of type (I'); it must, in particular, therefore, be equal to that formed for the normal form (18):

$$F_1(1, P_2, \ldots, P_n;\ \Delta P_2, \ldots;\ \ldots);$$

that is, it must be a function of the P's and their differences. Hence the functions $P_r[u(\xi)]$, or, expressed in terms of x, $P_r(x)$, $r=2, \ldots, n$, where

$$P_r(x)=\frac{p_r(x)}{p_1(x)\,p_1(x-1)\ldots p_1(x-r+1)},$$

form a fundamental set of invariants.

On the assumption that $p_{n-1}(x)$ is not identically zero, we readily obtain as a second fundamental set of invariants the functions $\Sigma_r(x)$, $r=n, \ldots, n-2$, where

$$\Sigma_r(x)=\frac{\pi_r(x)}{\pi_1(x)\,\pi_1(x+1)\ldots\pi_1(x+r-1)}.$$

The case in which all the coefficients preceding $p_s(x)$ (except the coefficient of y_{x+n}) and that in which all the coefficients following $p_{n-m}(x)$ (except the coefficient of y_x) are identically zero, lead to the following results:

THEOREM IV. *If all the coefficients in* (A) *preceding* $p_s(x)$ (*except, of course, that of* y_{x+n}, *which is unity*) *are identically zero, the functions* $S_r(x)$, $r=s+1, \ldots, n$, *where*

$$S_r(x)=\frac{p_r(x)\,p_r(x-r)\ldots\,p_r(x-\overline{s-1}\,r)}{p_s(x)\,p_s(x-s)\ldots\,p_s(x-\overline{r-1}\,s)},$$

form a fundamental set of invariants.

If all the coefficients in (A) *following* $p_{n-m}(x)$ (*except, of course, that of* y_x, *which is different from zero*) *are identically zero, the functions* $\Sigma_r(x)$, $r=n, \ldots, m+1$, *where*

$$\Sigma_r(x)=\frac{\pi_r(x)\,\pi_r(x+r)\ldots\,\pi_r(x+\overline{m-1}\,r)}{\pi_m(x)\,\pi_m(x+m)\ldots\,\pi_m(x+\overline{r-1}\,m)},$$

form a fundamental set of invariants.

In the important cases in which $s=1$ *and* $m=1$ *the functions* $S_r(x)$ *and* $\Sigma_r(x)$ *reduce to* $P_r(x)$ *and* $\Pi_r(x)$ *respectively, where*

$$P_r(x)=\frac{p_r(x)}{p_1(x)\,p_1(x-1)\ldots\,p_1(x-\overline{r-1})}$$

and

$$\Pi_r(x)=\frac{\pi_r(x)}{\pi_1(x)\,\pi_1(x+1)\ldots\,\pi_1(x+\overline{r-1})}.$$

§ 6. *Fundamental Sets of Semi-Covariants.*

We determine now certain fundamental sets of semi-covariants of (A).

Let us consider, for the moment, a particular transformation of the form in question, namely, one in which $\lambda(x)$ is a constant λ_1:

$$x=x, \quad y_x=\lambda_1\eta_x.$$

From the relation between y_x and η_x, we have

$$y_x=\lambda_1\eta_x;\; y_{x+1}=\lambda_1\eta_{x+1};\; \ldots;\; y_{x+n}=\lambda_1\eta_{x+n}. \tag{24}$$

Now any function that is a semi-covariant must remain the same for every constant value of λ_1; hence it must be independent of λ_1. Obviously, the only way in which λ_1 can be eliminated from equations (24) is by the setting up of a function that is homogeneous and of degree zero in the y's; hence every semi-covariant is expressible in terms of the ratios

$$\frac{y_{x+n-1}}{y_{x+n}},\; \frac{y_{x+n-2}}{y_{n+x}},\; \ldots,\; \frac{y_x}{y_{x+n}}$$

and their differences, and the p's and their differences.

Among these ratios there exists one relation, namely, equation (A); hence

but $n-1$ of them are independent. Let us take as the independent ratios

$$\frac{y_{x+n-r}}{y_{x+n}} = a_r, \quad r = 1, \ldots, n, \quad r \neq i,$$

where i is any integer between one and n. Furthermore, all the differences of the ratios a_r can be expressed in terms of the a's themselves, or in terms of the a's and the p's by aid of equation (A). Hence every semi-covariant must be of the form

$$F(a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n;\ p_1, \ldots;\ \Delta p_1, \ldots;\ \ldots).$$

Some very simple functions of this form are the following:

$$p_1(x)\frac{y_{x+n-1}}{y_{x+n}}, \quad p_2(x)\frac{y_{x+n-2}}{y_{x+n}}, \quad \ldots, \quad p_n(x)\frac{y_x}{y_{x+n}},$$

and these we readily show to be semi-covariants. Transforming (A) by the transformation

$$x = x, \quad y_x = \lambda(x)\eta_x,$$

we obtain

$$\eta_{x+n} + p'_1(x)\eta_{x+n-1} + \ldots + p'_n(x)\eta_x = 0,$$

where

$$p'_r(x) = p_r(x)\frac{\lambda(x+n-r)}{\lambda(x+n)}, \quad (r = 1, \ldots, n).$$

From the relation between $p_r(x)$ and $p'_r(x)$, and that between y_x and η_x, we obtain

$$p_r(x)\frac{y_{x+n-r}}{y_{x+n}} = p'_r(x)\frac{\lambda(x+n)}{\lambda(x+n-r)}\cdot\frac{y_{x+n-r}}{y_{x+n}} = p'_r(x)\frac{\eta_{x+n-r}}{\eta_{x+n}};$$

hence the functions

$$p_r(x)\frac{y_{x+n-r}}{y_{x+n}} = c_r, \quad (r = 1, \ldots, n)$$

are semi-covariants.

Obviously, no one of the semi-covariants c_r, $r = 1, \ldots, n$, $r \neq i$ can be expressed in terms of the remaining c's and their differences. Moreover, every semi-covariant can be expressed in terms of the c's and their differences, and seminvariants and their differences, as we shall now show, and hence the c's form a fundamental set.

We have already proved that every semi-covariant is expressible in the form

$$F(a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n;\ p_1, \ldots;\ \ldots).$$

By means of the relation

$$\frac{y_{x+n-r}}{y_{x+n}} = \frac{c_r}{p_r(x)},$$

we may write F in the form

$$F_1(c_1, c_2, \ldots, c_{i-1}, c_{i+1}, \ldots, c_n;\ p_1, \ldots;\ \ldots).$$

Now let F_1 be any semi-covariant. It must, by definition, be equal to the same function formed for any equation derived from (A) by a transformation of the form under consideration; it must, in particular, be equal to that set up for the normal form (7):

$$F_1(c_1, c_2, \ldots, c_{i-1}, c_{i+1}, \ldots, c_n; P_2, \ldots, P_n; \ldots).$$

Hence every semi-covariant is expressible in terms of the functions c_r, $r=1, \ldots, n$, $r \neq i$, and seminvariants and their differences.

We state our result in the form of a theorem, as follows:

THEOREM V. *Any $n-1$ of the functions c_r, $r=1, \ldots, n$, where*

$$c_r = p_r(x) \frac{y_{x+n-r}}{y_{x+n}},$$

form a fundamental set of semi-covariants.

§ 7. *Fundamental Sets of Covariants.*

By a method differing so little from that used in the determination of semi-covariants that it seems unnecessary to state it, it can be proved that the fundamental sets of semi-covariants determined in § 6 are also fundamental sets of covariants. We have then the theorem:

THEOREM VI. *Any $n-1$ of the functions c_r, $r=1, \ldots, n$, where*

$$c_r = p_r(x) \frac{y_{x+n-r}}{y_{x+n}}$$

form a fundamental set of covariants.

PART II. TRANSFORMATIONS AND INVARIANTS CONNECTED WITH LINEAR HOMOGENEOUS FUNCTIONAL EQUATIONS.

§ 8. *Determination of the Group of Transformations.*

We turn now to a study of the general linear homogeneous functional equation

$$y_{\theta^{(n)}(x)} + p_1(x)\, y_{\theta^{(n-1)}(x)} + \ldots + p_{n-1}(x)\, y_{\theta(x)} + p_n(x)\, y_x = 0, \tag{B}$$

where

$$\theta^{(n)}(x) = \theta[\theta^{(n-1)}(x)]\,; \quad \theta^{(1)}(x) \equiv \theta(x),$$

$\theta(x)$ being a known function.* The procedure is similar to that in the case of the difference equation.

We determine first the most general transformation of the type

$$x = \bar{u}(\xi, \eta_\xi), \quad y_x = \bar{v}(\xi, \eta_\xi) \tag{25}$$

* It will be noticed that for $\theta(x) = x \pm 1$, we have the difference equation. The difference equation was treated separately and with greater detail because of the importance of the equation and the greater simplicity of the treatment.

that transforms every linear homogeneous functional equation (B) of order n into a functional equation of the same type and order.

For the sake of simplicity we consider first an equation of order one:

$$y_{\theta(x)} + p_1(x)\, y_x = 0.$$

By means of transformation (25), this becomes

$$\bar{v}[\phi(\xi), \eta_{\phi(\xi)}] + p_1[\bar{u}(\xi, \eta_\xi)]\, \bar{v}(\xi, \eta_\xi) = 0,$$

where $\phi(\xi)$ denotes what ξ becomes when x is changed to $x+1$.

We readily see that in order that this equation may be linear in η_ξ, $\bar{u}(\xi, \eta_\xi)$ must be a function of ξ alone and $\bar{v}(\xi, \eta_\xi)$ must be linear in η_ξ; in order that it may be homogeneous, $\bar{v}(\xi, \eta_\xi)$ must be of the form $\lambda(\xi)\eta_\xi$. With these restrictions on ū and $\bar{v}$, the equation becomes

$$\lambda[\phi(\xi)]\, \eta_{\phi(\xi)} + p_1[u(\xi)]\, \eta_\xi = 0, \tag{26}$$

and the transformation takes the form

$$x = u(\xi), \quad y_x = \lambda(\xi)\, \eta_\xi.$$

In order that equation (26) may be a functional equation of the given type, of order unity, it is evident that one of the two relations

$$\left.\begin{aligned} \phi(\xi) &= \theta(\xi), \\ \phi(\xi) &= \theta^{-1}(\xi)^* \end{aligned}\right\} \tag{27}$$

must exist. This restriction says that when x is changed to $\theta(x)$, ξ becomes either $\theta(\xi)$ or $\theta^{-1}(\xi)$; it also imposes a restriction on $u(\xi)$ which we now determine. From the relation $x = u(\xi)$ we obtain at once

$$\xi = u^{-1}(x).$$

Let us now replace x by $\theta(x)$ and then substitute for x its equivalent $u(\xi)$:

$$\phi(\xi) = u^{-1}[\theta(x)] = u^{-1}\{\theta[u(\xi)]\}.$$

Using this latter value for $\phi(\xi)$ in the first of equations (27), we have

$$u^{-1}\{\theta[u(\xi)]\} = \theta(\xi),$$

whence

$$\theta[u(\xi)] = u[\theta(\xi)]. \tag{28a}$$

Using the same value for $\phi(\xi)$ in the second of equations (27), we have

$$u^{-1}\{\theta[u(\xi)]\} = \theta^{-1}(\xi),$$

whence

$$\theta[u(\xi)] = u[\theta^{-1}(\xi)]. \tag{28b}$$

Consider now the n-th order equation (B). Since the transformation sought must change every linear homogeneous functional equation into another of the

* If $\theta(\xi)$ has more than one inverse, any inverse can be taken for $\theta^{-1}(\xi)$.

same type and order, the restrictions on the transformation (25) found necessary for the case $n=1$ are necessary also for the general case. That they are sufficient as well is readily verified.

The group character of these transformations and the existence of a subgroup composed of those transformations in which $u(\xi)$ satisfies relation (28a) can be readily shown by the method used in § 2.

We have, then, the following theorems:

THEOREM VII. *The most general point transformation that transforms every linear homogeneous functional equation of the form*

$$y_{\theta^{(n)}(x)} + p_1(x)\, y_{\theta^{(n-1)}(x)} + \ldots + p_{n-1}(x)\, y_{\theta(x)} + p_n(x)\, y_x = 0$$

into a functional equation of the same type and order is of the form

$$\left.\begin{aligned} x &= u(\xi), \\ y_x &= \lambda(\xi)\eta_\xi, \end{aligned}\right\} \qquad \text{(II)}$$

where $u(\xi)$ *satisfies one of the two relations*

$$\theta[u(\xi)] = u[\theta(\xi)],$$
$$\theta[u(\xi)] = u[\theta^{-1}(\xi)].$$

THEOREM VIII. *The transformations of type* (II) *in which* $u(\xi)$ *satisfies the relation*

$$\theta[u(\xi)] = u[\theta(\xi)]$$

form a subgroup of the total group, while those in which $u(\xi)$ *satisfies the relation*

$$\theta[u(\xi)] = u[\theta^{-1}(\xi)]$$

do not.

§ 9. *Fundamental Sets of Seminvariants, Invariants, Semi-Covariants and Covariants.*

Fundamental sets of seminvariants, invariants, semi-covariants and covariants of the functional equation (B) can be determined by methods similar to those used in determining the corresponding functions in connection with the difference equation (A). Indeed the methods to be used here differ so little from those employed in the case of the difference equation that it seems unnecessary to do more than merely write out the results. We state as proved, then, the following theorems:

THEOREM IX. *If every coefficient in* (B) *preceding* $p_s(x)$ (*except, of course, the coefficient of* $y_{\theta^{(n)}(x)}$, *which is unity*) *is identically zero, the functions*

$I_r(x)$, $r = s+1, \ldots, n$, *where*

$$I_r(x) = \frac{p_r(x)\, p_r[\theta^{(-r)}(x)] \ldots p_r[\theta^{(-\overline{s-1}\, r)}(x)]}{p_s(x)\, p_s[\theta^{(-s)}(x)] \ldots p_s[\theta^{(-\overline{r-1}\, s)}(x)]}$$

and

$$\theta^{(-i)}(x) = \theta^{-1}[\theta^{(-\overline{i-1})}(x)], \quad \theta^{(-1)}(x) \equiv \theta^{-1}(x),$$

form a fundamental set of seminvariants and also a fundamental set of invariants.

If every coefficient in (B) *following* $p_{n-m}(x)$ (*except, of course, that of* y_x, *which must be different from zero*) *is identically zero, the functions* $\Omega_r(x)$, $r = n, \ldots, m+1$, *where*

$$\Omega_r(x) = \frac{\pi_r(x)\, \pi_r[\theta^{(r)}(x)] \ldots \pi_r[\theta^{(\overline{m-1}\, r)}(x)]}{\pi_m(x)\, \pi_m[\theta^{(m)}(x)] \ldots \pi_m[\theta^{(\overline{r-1}\, m)}(x)]}$$

and

$$\theta^{(i)}(x) = \theta[\theta^{(i-1)}(x)], \quad \theta^{(1)}(x) \equiv \theta(x),$$

form a fundamental set of seminvariants and also a fundamental set of invariants.

THEOREM X. *Any* $n-1$ *of the functions* $c_r(x)$, $r = 1, \ldots, n$, *where*

$$c_r(x) = p_r(x)\, \frac{y_{\theta^{(n-r)}}(x)}{y_{\theta^{(n)}}(x)},$$

form a fundamental set of semi-covariants and also a fundamental set of covariants.

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